The Ramsey Discounting Formula for a Hidden-State Stochastic Growth Process

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What is Approach of This Paper?

- Increasing fuzziness of distant-future growth rates as key issue.
- No one knows how to model fuzzy distant-future growth rates.
- Worth trying different approaches.
- Here try simplest Muth-Kalman hidden-state approach.
- Model here is a hybrid offspring of two parents.
- "Mother" = Ramsey formula connecting growth rates to discount rates.
- "Father" = Muth-Kalman hidden-state model of stochastic growth rates.
- Purposely pick specifications and functional forms to obtain *simple* analytical formulas.
- Hope that simple understandable formulas give useful general insights.

Basic Growth Model With a Visible State

$$\ln C_t - \ln C_{t-1} = Y_t \tag{1}$$

$$Y_t = X_t + z_t, \;\; z_t \sim iid\mathcal{N}(0, \; V_y) \tag{2}$$

$$X_t = X_{t-1} + w_t, \;\; w_t \sim iid\mathcal{N}(0, \; V_x) \tag{3}$$

$$\ln C_t - \ln C_0 = \sum_{\tau=1}^{t} \left(z_\tau + X_0 + \sum_{s=1}^{\tau} w_s \right) \tag{4}$$

$$\ln C_t - \ln C_0 \sim \mathcal{N}(tX_0, \; tV_y + t^3 V_x /3) \tag{5}$$

- In (5), where is the term $t^3 V_x /3$ coming from? Why cubic in time? What does it mean? What happens if X_0 is a random variable that is not observed directly?

Hidden-State Growth Model (**Quick** Version)

$$X_{t-1} \sim \mathcal{N}(\mu_{t-1}, V_{xy}) \tag{6}$$

$$X_{t-1} \to X_t \implies V_{xy} \to V_{xy} + V_x \tag{7}$$

$$Y_{t-1} \to Y_t \implies V_{xy} + V_x \to V_{xy} \implies \frac{1}{V_{xy} + V_x} + \frac{1}{V_y} = \frac{1}{V_{xy}} \tag{8}$$

$$\implies V_{xy} = \frac{\sqrt{4 V_x V_y + V_x^2} - V_x}{2} \approx \sigma_x \sigma_y \tag{9}$$

$$\left\{ \mu_0 = (1-\lambda) \sum_{s=0}^{\infty} \lambda^s Y_{-s}, \;\; \lambda = \frac{V_y}{V_{xy} + V_x + V_y} \right\} \tag{10}$$

$$t\, X_0 \sim \mathcal{N}(t\, \mu_0, t^2\, V_{xy}). \tag{11}$$

$$\ln C_t - \ln C_0 \sim \mathcal{N}\left(\mu_0\, t\, ,\; V_y\; t + \; V_{xy} t^2 + \frac{V_x}{3} t^3 \right). \tag{12}$$

Ramsey Hidden-State Discount Rates

$$W = E\left[\sum_{t=0}^{\infty} e^{-\rho t}\, U(C_t)\right] \tag{13}$$

$$\exp(-r_t\, t) = \frac{e^{-\rho t} E[U'(C_t)]}{U'(C_0)} \tag{14}$$

$$CRRA \implies U'(C) = C^{-\eta} \tag{15}$$

- Combine (12), (14), (15) and make use of formula for expectation of lognormal to obtain basic hidden-state formula

$$r_t = \rho + \eta\, \mu_0 - \frac{\eta^2}{2}\left(V_y + V_{xy}\, t + \frac{V_x}{3} t^2\right). \tag{16}$$

- Interpretation? Note linear and quadratic time decline in (16). Note that discount rate *eventually* becomes negative (meaning and significance?). Note effect of $V_y >> V_x \implies V_y >> V_{xy} >> V_x$.

Interpreting Ramsey Hidden-State Discounting Formula by Building up Sub-Components

$$V_y > 0,\ V_x > 0 : r_t = \rho + \eta\,\mu_0 - \frac{\eta^2}{2}\left(V_y + V_{xy}\,t + \frac{V_x}{3}t^2\right) \tag{17}$$

$$(V_y = 0,\ V_x = 0) \implies r_t = \rho + \eta\,\mu_0 \tag{18}$$

$$(V_y > 0,\ V_x = 0) \implies r_t = \rho + \eta\,\mu_0 - \frac{\eta^2}{2}V_y \tag{19}$$

$$(V_y = 0,\ V_x > 0) \implies r_t = \rho + \eta\,\mu_0 - \frac{\eta^2}{6}V_x\,t^2 \tag{20}$$

- Thought experiment: past is same but in *future* all uncertainty miraculously ceases (effectively X_t frozen at X_0). Then

$$r_t = \rho + \eta\,\mu_0 - \frac{\eta^2}{2}V_{xy}\,t \tag{21}$$

- What is interpretation of equation (21)?

A Numerical Example

- Immense subjectivity. Set $\rho = 0$, $\eta = 2$. Set $\mu_0 = 2\%$, $\sigma_y = 3\%$ (per year).
- Most brazen calibration is σ_x. I calibrate σ_x by requiring that the probability of a stagnant (no growth) century due to the random walk alone is one out of a million $(10^{-6}) \implies \sigma_x = .08\%$.
- With above parameter values, following table gives discount rate schedule

$t =$	0 yrs	50 yrs	100 yrs	150 yrs	200 yrs	250 yrs
$r_t =$	3.8%	3.5%	2.9%	2.1%	1.2%	.4%

Table 1: Discount rates r_t (% per year) as function of time t (years)

Concluding Questions and Comments

- What do we learn from all of this about long-term discounting?
- Three "source types" of lower discount rates under uncertainty.
- Two "source types" of time-declining discount rates.
- Seemingly insightful interaction between variances and time.
- Hint that forces causing declining discount rates may be powerful over long term. Even small amount of random walking can have eventual discounting impact over the long term.
- Sense that fuzziness about future growth rates is important ingredient in analysis.
- As yet unresolved how best to model fuzzy distant-future growth.
- What about caveats, limitations? Are alternative models "better"?
- More work is needed.